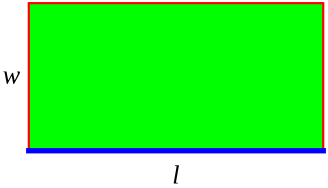
Constrained optimization

Problem: Design a fence to enclose a rectangular region of area 1200 m^2. Material for one edge (facing the street) costs \$50 per meter while material for the other three edges costs \$30 per meter.



Objective: Minimize $C = 50l + 30w + 2 \cdot 30w = 80l + 60w$.

Constraint: Need $lw = 1200$.

Method 1

Idea: Solve constraint for one of the variables and then substitute into the objective function to reduce the number of variables.

$$l = \frac{1200}{w} \quad \text{so} \quad C = 80l + 60w = 80\frac{1200}{w} + 60w = \frac{96000}{w} + 60w.$$

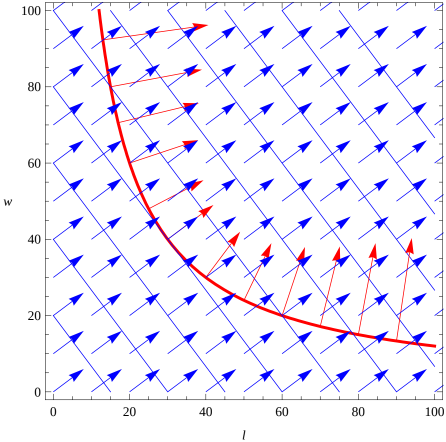
Compute $C' = -\frac{96000}{w^2} + 60.$

Solve $-\frac{96000}{w^2} + 60 = 0$ to get $w = \pm 40.$

Use $w = 40$ to get $l = \frac{1200}{40} = 30$

So build fence with expensive edge of length 30 meters and other dimension of 40 meters.

Idea for Method 2



Constraint curve $A = lw = 1200$

Level curves for objective $C = 80l + 60w$

Gradient vectors for constraint $A = lw$

Gradient vectors for objective $C = 80l + 60w$

Idea for Method 2

Maximum or minimum of objective along constraint curve will be at a point where

objective level curve is tangent to constraint curve

$$\Updownarrow$$

objective gradient $\vec{\nabla} C$ is aligned with constraint gradient $\vec{\nabla} A$

$$\Updownarrow$$

$\vec{\nabla} C = \lambda \vec{\nabla} A$ for some constant λ